Bond algebras and exact solvability of Hamiltonians: Spin $S=\frac{1}{2}$ multilayer systems

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We present an algebraic methodology for designing exactly solvable Lie model Hamiltonians. The idea consists in looking at the algebra generated by *bond operators*. We illustrate how this method can be applied to solve numerous problems of current interest in the context of topological quantum order. These include Kitaev's well-known toric code and honeycomb models as well as new models: a vector-exchange model and a Clifford γ model in a triangular lattice.

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I. INTRODUCTION

Whenever one is interested in studying a new physical phenomena whose effective model includes degrees of freedom (spins, fermions, bosons, etc.) which are strongly coupled, one attempts to invoke simplifying assumptions hoping that the resulting problem represents the relevant minimal model for the phenomenon at hand. Finding exactly solvable models is always welcome. This paper is about a general methodology to generate exactly solvable Hamiltonians by concentrating on the algebra generated by algebraic objects called *bonds*. These objects refer to the basic terms in the Hamiltonian (which can be the usual two-particle *bonds* or more complicated interactions involving more elementary degrees of freedom—the method that we will present in the current work applies to general interactions). We have already used this methodology in Ref. 1, where we solved a doped orbital compass model in two and three space dimensions, although we did not explain the generality of the mathematical approach. A goal of this paper is to present this methodology in full detail and show that by using the algebra of bond operators we can easily construct whole families of exactly solvable models, several of these displaying topological quantum order.[2] The power of our method is that it enables mappings between systems in various space (or space-time) dimensions. Our method renders the investigation of many systems in arbitrary dimensions and different arenas far simpler than has been realized hitherto and enables the investigation of other systems.

For the sake of clarity, we will focus on quantum lattice systems which have $N_s=\Pi_{\mu=1}^{D}L_\mu$ sites, with L_μ as the number of sites along each spatial direction μ and D as the dimensionality of the lattice. Unlike many beautiful tools that have great applicability in one-dimensional (1D) quantum models (e.g., Bethe ansatz, Bosonization, and quasiexact solvability),[3] our results will apply to systems in general spatial dimensions $D\geq 1$. The connectivity of the D dimensional lattice and its general *topology* are of paramount importance. Associated with each lattice site $\mathbf{i}\in\mathbb{Z}^{N_s}$ there is a Hilbert-space $\mathcal{H}_\mathbf{i}$ of finite dimension $\mathcal{D}_\mathbf{i}$. The total Hilbert space is the tensor product of the local state spaces, $\mathcal{H}=\otimes_\mathbf{i}\mathcal{H}_\mathbf{i}$, in the case of distinguishable subsystems (or a proper subspace in the case of indistinguishable ones), and its dimension is $\mathcal{D}=\Pi_{\mathbf{i}=1}^{N_s}\mathcal{D}_\mathbf{i}$. The dynamics of an arbitrary physical system is governed by a Hamiltonian H whose form is constrained, in terms of the local language,[4] to linear combinations of (polynomial in N_s) quasilocal operators.

Let us first start with an intuitive introduction to the key concept of *bond algebras*. We consider situations in which the Hamiltonian of a system H, whose state space is $\mathcal{H}$, can be written as a sum of *quasilocal* terms or *bonds*

$$H=\sum_R \alpha_R h_R, \tag{1}$$

where α_R is a c number, $\{h_R\}$ are the bond operators, and R includes a finite number of, for example, lattice sites $\mathbf{i}$. In general, the operators h_R will generate a certain (*bond*) algebra $\mathcal{G}$ whose dimension is of order $\mathcal{D}$, i.e., $O(\mathcal{D})$. To simplify the description, in the following, we are going to concentrate on semisimple Lie algebras. Notice that we do not constrain ourselves to a particular representation of the algebra.

It may happen that the Hamiltonian, H, itself is an element of a subalgebra (denoted by $\mathfrak{h}$) of the algebra $\mathcal{G}$ and that this subalgebra is of dimension poly-log $\mathcal{D}$ (i.e., polynomial in the logarithm of the Hilbert-space dimension). In other words, H may be a (polynomial in log $\mathcal{D}$) linear combination of operators h_R [see Eq. (1)] that form a subalgebra $\mathfrak{h}$ (of polynomial in log $\mathcal{D}$ dimension). Those Hamiltonians are called *generalized mean-field Hamiltonians* (GMFHs), and traditional examples include Hartree-Fock, Random-Phase, and Bardeen-Cooper-Schrieffer Hamiltonians, while less known examples may even involve superalgebras.[5] These GMFHs are mean field in the sense that they represent effective *one-particle* Hamiltonians. When the ground state of H is nondegenerate, it turns out to be a generalized coherent state of $\mathfrak{h}$,[5] while the remaining eigenstates (some of which may also be generalized coherent states) and energies can be efficiently computed.

We say that H is *exactly solvable* when an arbitrarily chosen eigenvalue and an appropriate description of the corresponding eigenstate can be obtained and represented to precision ϵ by means of a classical algorithm efficient in log $\mathcal{D}$ and $1/\epsilon$.[5] This definition of exact solvability, motivated by complexity theory, yields a sufficient criterion for exact solvability and is general enough to also include Bethe-ansatz solvable problems. In an algebraic language, Bethe-ansatz systems are also characterized by Lie algebraic Hamiltonians

but those are not elements of a polylog $\mathcal{D}$-dimensional Lie subalgebra, so they are not GMFHs. One can show that if H is a GMFH then it is exactly solvable, and there is a polynomially in log $\mathcal{D}$-efficient algorithm to diagonalize it[5] (this Jacobi-like method is described in Ref. 5). Notice that the α_R coefficients in Eq. (1) can be arbitrary (e.g., a disordered system) with no particular space-group symmetry being respected. In other words, what is relevant for exact solvability is the algebraic structure of H. A particular case of exact solvability is when the spectrum can be expressed in closed form.

It is important to emphasize that the concept of exact solvability above refers to the computation of *an arbitrary* eigenvalue of H (and its corresponding eigenvector). In statistical mechanics, the concept of exact solvability typically refers to the complexity of determining the *partition function*, $\mathcal{Z}$, of a system. As is well-known, the partition function and the density of states associated with the spectrum are related by a Laplace transform. It turns out that the exact solvability of H does not imply that $\mathcal{Z}$ can be determined with poly-log $\mathcal{D}$ complexity, i.e., that $\mathcal{Z}$ is exactly solvable. For example, Kitaev's toric code and honeycomb models[6,7] are exactly solvable Hamiltonians. On the other hand, Kitaev's toric model $\mathcal{Z}$ can be exactly determined (as shown in Refs. 8 and 9), while Kitaev's honeycomb model $\mathcal{Z}$ cannot. Perhaps a more standard example is the classical Ising model: in a square lattice both H and $\mathcal{Z}$ are exactly solvable, while on a cubic lattice only H can be exactly solved.

In the following we will show that if two systems (whether they are exactly solvable or not) display the same bond algebra $\mathcal{G}$ and representation, then their spectra are identical. A main contribution of this paper is to propose a methodology to generate exactly solvable Hamiltonians by using two mathematical principles that will become evident in the following sections. In all cases, this methodology rests on (1) topological constraints that are related to the connectivity of the lattice Hamiltonian or graph. In several instances, it further relies on the existence of (2) gauge symmetries. These symmetries allow a decomposition of the Hilbert space into sectors. The operators $\{h_R\}$ belong to the lowest-dimensional representation of the algebra on the Hilbert space, or its subspaces. In the following we illustrate the bond algebra methodology by showing some tutorial examples of known trivially *exactly solvable* problems.

A. Ising model

A simple example is afforded by the Ising model on a hypercubic lattice of N_s sites,

$$H_{\text{Ising}} = -\sum_{\langle ij \rangle} J\sigma_i\sigma_j. \tag{2}$$

The bonds $b_{ij} \equiv \sigma_i\sigma_j$ satisfy a simple Ising (Abelian)-type algebra defined on a $\mathcal{D}=2^{N_s}$-dimensional space (the span of the original Ising system)

$$[b_{ij}, b_{kl}] = 0, \quad b_{ij}^2 = 1, \tag{3}$$

since $\sigma_i = \pm 1$. All classical Hamiltonians are extreme cases of GMFHs: Its spectra are trivially determined.[10]

B. Transverse field Ising chain

The Hamiltonian of a single transverse field Ising chain of length N_s reads

$$H_{\text{TFIM}} = -\sum_{i=1}^{N_s} (J_i\sigma_i^y\sigma_{i+1}^y + h_i\sigma_i^x), \tag{4}$$

where $\sigma_i^\mu (\mu = x,y,z)$ represent Pauli matrices. To make clear the algebraic connection that will follow, let us denote the two terms (transverse field and bond variables) as follows:

$$\bar{A}_i^x = \sigma_i^x, \quad \bar{A}_{i,j} = \sigma_i^y\sigma_j^y. \tag{5}$$

In terms of these, the Hamiltonian of Eq. (4) obviously reads

$$H_{\text{TFIM}} = -\sum_{i=1}^{N_s} (J_i\bar{A}_{i,i+1} + h_i\bar{A}_i^x), \tag{6}$$

with interaction terms satisfying

$$[\bar{A}_i^x, \bar{A}_j^x] = 0 = [\bar{A}_{i,j}, \bar{A}_{k,l}],$$

$$\{\bar{A}_i^x, \bar{A}_{i,i+1}\} = 0 = \{\bar{A}_i^x, \bar{A}_{i-1,i}\},$$

$$[\bar{A}_i^x, \bar{A}_{j,k}] = 0, \quad i \neq j,k,$$

$$(\bar{A}_i^x)^2 = 1 = (\bar{A}_{j,k})^2, \tag{7}$$

which forms an so($2N_s$) (poly-log $\mathcal{D}$) algebra with $\mathcal{D}=2^{N_s}$. More explicitly, the number of generators of so($2N_s$) is $N_s(2N_s-1)$, i.e., $\mathcal{O}(N_s^2)$.

Note that the bond algebra encapsulated in the relations above is invariant under the flip of any transverse field locally. The transformation

$$\bar{A}_i^x \to -\bar{A}_i^x \tag{8}$$

effects $h_i \to -h_i$ at the lattice site i. Indeed, all that a flip of local fields does is to leave the spectrum unaltered while permuting the eigenstates among themselves. In more conventional terms, the invariance of the spectrum mandated by the invariance of the bond algebra under the transformation of Eq. (8) is seen by noting that a similarity transformation with the local unitary (and Hermitian) operator $U_i = \sigma_i^y$ sets

$$\sigma_i^y\sigma_i^x\sigma_i^y = -\sigma_i^x, \tag{9}$$

while leaving σ_i^y and thus, $\bar{A}_{i,i+1}$ invariant. The spectrum of Eq. (6) can be determined by performing a Jordan-Wigner transformation to free fermions. Equivalently, it may be noted that the bond algebra of a tight-binding spinless Fermi model (with pairing terms) is equivalent to that of Eq. (7).

C. Orbital compass chain model

This model was introduced in Ref. 11. It consists of a $D=1$-dimensional system with alternating xx and yy interactions. Namely, consider a chain of length N_s in which the Hamiltonian is given by

$$H_{\mathrm{OCM}} = \sum_{i=1,3,5,\cdots} J_{x,i}\sigma_i^x\sigma_{i+1}^x + \sum_{i=2,4,6,\cdots} J_{y,i}\sigma_i^y\sigma_{i+1}^y. \quad (10)$$

Let us define the even and odd bonds by

$$A_m = \sigma_{2m}^y\sigma_{2m+1}^y, \quad B_m = \sigma_{2m-1}^x\sigma_{2m}^x. \quad (11)$$

They satisfy the following algebra ($\mathcal{D}=2^{N_s}$):

$$[A_m,A_n] = 0 = [B_m,B_n]$$

$$\{A_m,B_m\} = 0 = \{A_m,B_{m+1}\},$$

$$[A_m,B_n] = 0, \quad |m-n| > 1$$

$$(A_m)^2 = 1 = (B_m)^2. \quad (12)$$

This algebra is identical to the algebra of bonds of Eq. (7). For an open chain, there are no boundary conditions on the bonds in either problem. If we enabled interactions (both exchange and transverse fields) on only one half of the chain (that is, if the sum in Eq. (4) would extend, for even N_s, only from $1 \le i \le N_s/2$) and add $N_s/2$ noninteracting spins, then the number of interaction terms in Eqs. (4) and (10), their algebras (and dimension of their representations), and the size of the Hilbert space on which both systems are defined are identical. In that case, the partition functions are identical up to a trivial multiplicative factor (after identifying $J_i=J_{y,i}$ and $h_i=J_{x,i}$)

$$\mathcal{Z}_{\mathrm{OCM}}(N_s) = 2^{N_s/2}\mathcal{Z}_{\mathrm{TFIM}}(N_s/2). \quad (13)$$

Such a relation was indeed found by[11] an explicit diagonalization of the Fermi bilinear found after a Jordan-Wigner transformation performed on H_{OCM}. Here we arrived at the same result by a trivial application of the methodology of bond algebras.

To attest to the power of our method and its applications, it is worth noting that the bond algebraic mapping gives rise to an immediate corollary. As the equations of motion in a general system are determined by the algebra (e.g., the commutators of the operators with the Hamiltonian), two systems that share the same bond algebra must also share the same dynamics. Thus, the dynamics of the orbital compass chain must be equivalent to that of the bonds within the transverse field Ising model. This immediately gives rise to the dynamics in this system without the need to perform anew a very detailed study. Indeed, Ref. 12 found that the dynamics of the orbital compass chain is that of the transverse field Ising model.

D. Kitaev's toric code model

Kitaev's toric code model[6] is defined on a square lattice with $L\times L=N_s$ sites, where on each bond (or link) (ij) is an $S=1/2$ degree of freedom indicated by a Pauli-matrix σ_{ij}^{μ}. The Hamiltonian acting on a $\mathcal{D}=2^{2N_s}$-dimensional Hilbert space is

$$H_K = -\sum_s A_s - \sum_p B_p, \quad (14)$$

with Hermitian operators (whose eigenvalues are ± 1)

$$A_s = \prod_{(ij)\in \mathrm{star}(s)} \sigma_{ij}^x, \quad B_p = \prod_{(ij)\in \mathrm{plaquette}(p)} \sigma_{ij}^z, \quad (15)$$

where B_p and A_s describe the plaquette (or face) and star (or vertex) operators associated with each plaquette p, and each site s of the square lattice. The reader may want to consult Refs. 8, 9, and 13 for notation purposes.

That the $D=2$ Kitaev's toric code model is identical to two decoupled Ising chains[8,9] is immediately seen by looking at the bond algebra. The algebra of the bonds given by Eq. (15) is trivial, it is an Ising (Abelian)-type algebra

$$[A_s,B_p] = [A_s,A_{s'}] = [B_p,B_{p'}] = 0,$$

$$(A_s)^2 = 1 = (B_p)^2. \quad (16)$$

For periodic boundary conditions one has the additional constraint

$$\prod_s A_s = \prod_p B_p = 1. \quad (17)$$

It is very easy to realize that the Hamiltonian for two decoupled Ising chains, each of length N_s

$$H_I = -\sum_{s=1}^{N_s} \sigma_s\sigma_{s+1} - \sum_{p=1}^{N_s} \tau_p\tau_{p+1}, \quad (18)$$

with $\sigma_s=\pm 1$ and $\tau_p=\pm 1$, displays an identical bond algebra to Eq. (16), with the same representation. Thus, one can immediately write down the partition function[8,9,13]

$$\mathcal{Z}_K = (2\cosh\beta)^{2N_s}(1+\tanh^{N_s}\beta)^2, \quad (19)$$

where $\beta=1/(k_BT)$, and T is temperature. Moreover, the bond algebra of Kitaev's toric code model is identical to that of Wen's plaquette model[14] which proves the equivalence of the two systems.[9] It is worthwhile to note that Eq. (19) is also the outcome of a high-temperature series expansion.[15]

Thus, Kitaev's toric code model is identical to a one-dimensional Ising system. This statement has ramifications for the stability of quantum memories—an item that we investigated in detail early on.[8,9,13] This mapping allows not only an evaluation of the partition function but also a direct computation of all correlators in Kitaev's toric code model. For a detailed explanation see Refs. 9 and 13. In particular, see subsections (XIII A,B) as well as footnotes 61–63 of Ref. 9. The equations of motion with uncorrelated noise are insensitive to a change in basis. Consequently, the dynamics and thermal effects present in one-dimensional systems rear their head also in Kitaev's toric code model. In particular, the system is unstable to thermal noise—a phenomenon that we coined *thermal fragility*[8,9,13] and has been recently confirmed by others.[16–19] Our bond algebraic mapping enables an immediate extraction of crossovers for finite-size systems.[20–22]

E. Plaquette model in a transverse magnetic field

This model[23] is defined on a square lattice as follows:

$$H_{\square} = -\sum_i J_i F_i - \sum_i h_i \sigma_i^x, \tag{20}$$

where $F_i = \sigma_i^x \sigma_{i+\hat{e}_x}^y \sigma_{i+\hat{e}_x+\hat{e}_y}^x \sigma_{i+\hat{e}_y}^y$, with $\hat{e}_\mu$ representing unit vectors along the μ direction in the lattice.

Setting $G_i \equiv \sigma_i^x$ and $F_i \equiv F_{i*}$ with $i* \equiv i + \frac{1}{2}(\hat{e}_x + \hat{e}_y)$ and a diagonal chain coordinate j along the (1,1) direction, that alternates between i and $i*$, the algebra of the interaction terms (*bonds*) in Eq. (20) is

$$[F_j, F_{j'}] = 0 = [G_j, G_{j'}]$$

$$\{F_j, G_{j+1}\} = 0 = \{F_j, G_{j-1}\},$$

$$[F_j, G_{j'}] = 0 \left(j - j' \neq \pm \frac{1}{2}(\hat{e}_x + \hat{e}_y) \right)$$

$$(F_i)^2 = 1 = (G_i)^2. \tag{21}$$

For a system with open boundary conditions, the algebra of this system is none other than that of a stack of decoupled transverse field Ising chains [see Sec. I B and Eq. (7) in particular] all oriented diagonally along the (1,1) direction. Setting, in Eq. (7), $\bar{A}_{ij} \equiv \bar{A}_{i*}$, we see that the algebra and the dimension of the Hilbert space in both problems are identical. Indeed, a more elaborate treatment finds that this system is none other than that of a transverse field Ising model[23] precisely as we find by examining the bond algebra. The partition function is, therefore, exactly the same as that of a transverse field Ising model.

In the next sections, we illustrate the power of our method by reviewing several, more challenging, known examples of exactly solvable models whose solutions can be immediately achieved in this way, and then we turn to new models that we introduced and solved using these tools. We start by discussing Kitaev's honeycomb model[7] and show that no enlargement of the Hilbert space[7] nor a direct Jordan-Wigner mapping[24] is necessary to solve this model in a very short and direct manner. Next, we will turn to new models. The first of these is the *vector-exchange* model which forms a simple extension of Kitaev's honeycomb model. We will later on show that all these models have in common a Clifford algebraic structure on-site and an Abelian structure off-site. This defines a simple class of GMFH of the so($2N$) type.[5] More general Lie algebraic structures can be also realized.

II. KITAEV'S HONEYCOMB MODEL

A. Spectrum from bond algebras

Kitaev's honeycomb lattice model[7,25] is a member of a family of models whose Hamiltonians are elements of the so($2N_s$) algebra, where N_s is the number of vertices of the honeycomb lattice, i.e., it is a GMFH.

The model is defined by the following $S=1/2$ Hamiltonian (Fig. 1):

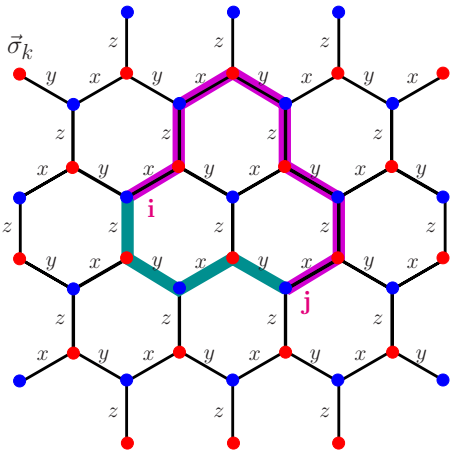

FIG. 1. (Color online) Kitaev's model on a honeycomb lattice and three types of bonds. On each vertex there is an $S=1/2$ degree of freedom indicated by a Pauli-matrix $\vec{\sigma}_k$. There are two different types of vertices. Thick colored contours represent arbitrary paths drawn on the lattice, e.g., from site i to j.

$$H_{K_h} = -J_x \sum_{x-\text{bonds}} \sigma_i^x \sigma_j^x - J_y \sum_{y\ \text{bonds}} \sigma_i^y \sigma_j^y - J_z \sum_{z\ \text{bonds}} \sigma_i^z \sigma_j^z$$
$$= -\sum_{\langle ij \rangle} J_{ij} \sigma_i^\mu \sigma_j^\mu \quad [\hat{e}_\mu \| (\vec{j} - \vec{i})]. \tag{22}$$

Let us define the bond operators

$$A_{ij}^\mu = \sigma_i^\mu \sigma_j^\mu, \quad \mu = x,y,z, \tag{23}$$

where $\langle i,j \rangle$ defines a bond (nearest neighbors). We can assign a uniform direction to all bonds by choosing for all nearest-neighbor sites i and j, the bond defined by A_{ij}^μ corresponds to $j-i=\hat{e}_\mu$ (and not $j-i=-\hat{e}_\mu$). From the commutation relations for SU(2) spins it is clear that ($\Delta_{ijkl} = \delta_{ik} + \delta_{jk} + \delta_{il} + \delta_{jl}$)

$$[A_{ij}^\mu, A_{kl}^\nu] = (1 - \delta_{\mu\nu})(1 - (-1)^{\Delta_{ijkl}}) A_{ij}^\mu A_{kl}^\nu, \tag{24}$$

and, moreover, it is clear that $(A_{ij}^\mu)^2 = 1$. For an arbitrary set of bond operators A_{ij}^μ, the Lie algebra $\mathcal{G}$ generated is $O(\mathcal{D})$. However, the set of bond operators that appear in H_{K_h}, $\{A_{ij}^\mu\}_{H_{K_h}}$, forms a Lie subalgebra of $\mathcal{G}$, because of the particular lattice topology: bond operators that share a vertex anticommute, otherwise they commute. This subalgebra is precisely so($2N_s$), and the Hamiltonian being an element of that subalgebra is an GMFH.

One could, in principle, stop here and diagonalize the problem in a Hilbert space of dimension $\mathcal{D} = 2^{N_s}$. However, there is a further simplification in this problem. The simplification is related to exploiting the existence of *gauge symmetries*. Consider the *anyon charge*[7] operators

$$I_{h_\alpha} = \prod_{\langle ij \rangle \in h_\alpha} A_{ij}^\mu = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z,$$

where h_α defines a particular hexagonal plaquette (see Fig. 1). The $\{I_{h_\alpha}\}$ operators have eigenvalues $\tilde{I}_{h_\alpha} = \pm 1$, and they satisfy the following relations:

$$[I_{h_\alpha}, I_{h_{\alpha'}}] = 0, \quad (I_{h_\alpha})^2 = 1,$$

$$[I_{h_\alpha}, A_{ij}^{\mu}] = 0, \tag{25}$$

which implies that $[I_{h_\alpha}, H_{K_h}]=0$. In other words, the product of bonds taken around the hexagon h_α in a uniform orientation (either clockwise or counterclockwise) is a gauge symmetry. The $N_s/2$ operators $\{\tilde{I}_{h_\alpha}\}$ satisfy the global constraint

$$\prod_{h_\alpha} \tilde{I}_{h_\alpha} = 1. \tag{26}$$

This allows us to decompose the $\mathcal{D}$-dimensional Hilbert-space $\mathcal{H}$ into $2^{N_s/2-1}$ orthogonal Hilbert subspaces $\mathcal{H}_\eta$, each of dimension $\dim(\mathcal{H}_\eta)=2^{N_s/2+1}$

$$\mathcal{H} = \bigoplus_{\eta=1}^{2^{N_s/2-1}} \mathcal{H}_\eta. \tag{27}$$

Each Hilbert subspace $\mathcal{H}_\eta$ is characterized by a particular set of eigenvalues $\{\tilde{I}_{h_\alpha}\}$ and projector

$$\hat{P}_\eta = \prod_{\alpha=1}^{N_s/2} \frac{\mathbb{1} + \tilde{I}_{h_\alpha} I_{h_\alpha}}{2} = \prod_{\alpha=1}^{N_s/2} \hat{P}_{I_{h_\alpha}}. \tag{28}$$

The algebra satisfied by the projected bond operators $\bar{A}_{ij}^{\mu} = \hat{P}_\eta A_{ij}^{\mu} \hat{P}_\eta$ is so$(2N_s)$, but acts on a Hilbert (carrier) subspace of dimension $2^{N_s/2+1}$

To determine the spectrum in each subspace we look for an *oscillator realization* of the algebra

$$\bar{A}_{ij}^{\mu} = 2i\eta_{ij} c_i c_j, \quad \mu = x,y,z, \tag{29}$$

in terms of Majorana fermions c_i, which satisfy

$$\{c_i, c_j\} = \delta_{ij}, \quad c_i^\dagger = c_i. \tag{30}$$

The smallest representation of N_s Majorana fermion modes (N_s even) is that in a $2^{N_s/2}$-dimensional Hilbert space. For reasons that will become clear later on, in Eq. (29) we will set $\eta_{ij}=1$ on all bonds parallel to the "x" or "y" directions and allow $\eta_{ij}=\pm 1$ on all vertical bonds (those parallel to the "z" direction in which $\vec{j}-\vec{i}=\pm\hat{e}_z$). There remains an additional degeneracy factor of two (the Hilbert-space dimension is of size $2^{N_s/2+1}$ while the representation of the bonds is on a Hilbert space of size $2^{N_s/2}$).

It is straightforward to show that the bilinear combinations of Majorana fermions satisfy the same algebra and also constraints as the algebra of $\{A_{ij}^{\mu}\}_{H_{K_h}}$. Notice that any connected open string product of bonds becomes a bilinear in Majorana fermions[24] (see Fig. 1)

$$S_{i_1,i_L} = \bar{A}_{i_1 i_2}^{\mu_1} \bar{A}_{i_2 i_3}^{\mu_2} \cdots \bar{A}_{i_L i_{L+1}}^{\mu_L} = 2i^L \left(\prod_{\langle i,j \rangle} \eta_{ij} \right) c_{i_1} c_{i_{L+1}}. \tag{31}$$

It turns out that all open strings having the same end points i_1, i_L and with alternating $\mu's$ (e.g., x,z,y,z,y,x,z,x) can be of only four types

$$S_{i_1,i_L} = (\pm 1, \pm i) 2 c_{i_1} c_{i_{L+1}}, \tag{32}$$

and form a polynomial in the number of vertices (or bonds) Lie algebra. The correspondence between the anyon charge sector $\{\tilde{I}_{h_\alpha}\}$ and the set $\{\eta_{ij}\}$ is

$$\prod_{\langle ij \rangle \in h_\alpha} \eta_{ij} = \tilde{I}_{h_\alpha}. \tag{33}$$

The set $\{h_\alpha\}$ spans all fundamental hexagonal plaquettes from which all closed-loops Γ can be uniquely constructed. The number of plaquettes $\{h_\alpha\}$ is given by half the number of sites $N_s/2$ as is the number of vertical bonds on which we may assign one of the two phases corresponding to $\eta_{ij} = \pm 1$. With the identification of Eq. (33), both sides of Eq. (29) satisfy the same set of algebraic relations in each of the $2\times 2^{N_s/2}$-dimensional Hilbert subspaces.

This mapping allows us to immediately write down the spectrum in each sector of $\{I_{h_\alpha}\}$ and to reproduce the results of Refs. 7 and 24 without the need for introducing two Majorana fermions per spin and then projecting out one (as in Ref. 7) nor writing an explicit Jordan-Wigner transformation between fermionic and the spin variables (as was done in Ref. 24).

In a given sector $\eta=\{\eta_{ij}\}$, we have the Majorana fermion representation of the Hamiltonian,

$$H_{K_h,\eta} = 2i \sum_{\langle ij \rangle} \eta_{ij} J_{ij} c_i c_j, \tag{34}$$

where $J_{ij}=J_x$ if i and j are separated by an x-type bond. Similarly, $J_{ij}=J_{y,z}$ if i and j are linked by a y- or z-type bond. Within the ground-state sector ($I_{h_\alpha}=1$ for all plaquettes h_α), we may set $\{\eta_{ij}=1\}$ and obtain the quasiparticle spectrum, $\vec{k}=(k_x,k_y)$,[7,24]

$$E_{\vec{k}} = \pm \sqrt{\epsilon_{\vec{k}}^2 + \Delta_{\vec{k}}^2},$$

$$\epsilon_{\vec{k}} = 2J_z - 2J_x \cos k_x - 2J_y \cos k_y,$$

$$\Delta_{\vec{k}} = 2J_x \sin k_x + 2J_y \sin k_y. \tag{35}$$

Our mapping allows for a closed-form solution only for a reduced set of sectors (such as the ground-state sector). For the rest, we still can compute each eigenvalue and eigenvector with polynomial in N_s complexity by using the Jacobi method.[5] Thus, it is not simple to compute the partition function of the model with the same complexity. One can write down a formal solution, as we discuss below, but it is not a closed-form analytical solution in terms of simple functions.

B. Partition function

Although many results appear on the zero-temperature behavior of Kitaev's honeycomb model, there are very few results at finite temperatures. An exception is Ref. 26 which provides a finite temperature metric analysis of Kitaev's honeycomb model. Related results are discussed in Ref. 13.

The partition function includes contributions from all sectors and reads[13]

$$\mathcal{Z} = 2^{N_h - N_s} \sum_{\eta} \mathcal{Z}_{\eta}$$

$$\mathcal{Z}_{\eta} = \mathrm{Tr}\, \exp[-\beta H_{K_h,\eta}], \tag{36}$$

where $N_h = \frac{N_s}{2}$ is the number of hexagonal plaquettes. In terms of the original spins of Eq. (22)

$$\mathcal{Z} = \mathrm{Tr} \sum_{n=0}^{\infty} \frac{(\beta H_{K_h})^{2n}}{(2n)!}, \tag{37}$$

or, equivalently, in terms of the Majorana representation of Eqs. (29) and (34)

$$\mathcal{Z}_{\eta} = \mathrm{Tr} \sum_{n=0}^{\infty} \frac{(\beta H_{K_h,\eta})^{2n}}{(2n)!}. \tag{38}$$

The reason why in Eq. (37) we keep only the even powers of H_{K_h} is the following: by time-reversal symmetry [due to the trace over σ_i^{μ} and $(-\sigma_i^{\mu})$], at any given site i we must have an even power of σ_i^{μ}. Similarly, in the Majorana fermion representation [Eq. (38)], the trace of c_i is zero. For any odd power of H_{K_h}, there is in any term resulting from the expansion of $\exp[-\beta H_{K_h}]$ at least one site for which we have an odd power of σ_i^{μ} (or c_i). All of these terms vanish.

Let us first consider the Majorana representation and focus on $\mathcal{Z}_{\eta}$. We will later on rederive these results within the original spin representation of Eq. (22). The local assignments $\{\eta_{ij}\}$ effectively relate J_{ij} in a general sector to that in the sector $\{\eta_{ij}=1\}$ by the transformation

$$J_{ij}\eta_{ij} \leftrightarrow J_{ij}. \tag{39}$$

We claim that if a particular bond (J_{ij}) appears as an odd power in a given term then it will give rise to a vanishing contribution when it is traced over. The proof of this assertion is trivial

$$\sum_{\eta_{ij}=\pm 1} \eta_{ij}^{p} J_{ij}^{p} = 0 \tag{40}$$

for all odd p.

The same conclusion follows within the original spin representation of Eq. (22) which as we show below leads to Eq. (37). Let us mark all the bonds J_{ij} that would additionally appear to an odd power in the expansion of Eq. (37). We claim that there are several possible topologies: (i) three odd bonds (odd powers of J_{ij}) meet at a common vertex. That is, we can have

$$J_{ij}^{p_{ij}} J_{ik}^{p_{ik}} J_{il}^{p_{il}}, \tag{41}$$

with odd $p_{ia}(a=j,k,l)$ and with all of the bonds that touch j, k, and l appearing to an even power in the expansion of Eq. (37).

(ii) Closed or open contours of odd powered bonds appear. In case (i), the spins at sites j, k, and l appear to an odd power (the power is just the sum of the powers of the bonds that have one of these points at their end). In case (ii), if the contour is open then the spins at the end points of the open contour must appear to an odd power and therefore lead to a term that vanishes upon taking the trace. If the contour is closed it leads to none other than the anyon charge within the contour C,

$$I_C = \prod_{h_{\alpha} \in C} I_{h_{\alpha}}, \tag{42}$$

with $I_{h_{\alpha}}$ the product of bonds along a hexagonal loop. Using the relation $\sigma^{\mu}\sigma^{\nu} = i\epsilon_{\mu\nu\kappa}\sigma^{\kappa}$ for $\mu \neq \nu$, we find that each spin i that lies on $C(i \in C)$ leads to a contribution σ_i^{κ}. For any odd power p, $[\sigma_i^{\kappa}]^p$ has a vanishing trace.

We can similarly have both (i) and (ii). It is readily seen that all odd powered bonds lead to situations with either odd powers of the spins at the end points and/or to closed contours which also lead to vanishing contribution.

Returning to the sum of Eqs. (36) and (38), we now have

$$\mathcal{Z} = 2^{N_h - N_s + 1} \mathbf{E}[\mathcal{Z}_1], \tag{43}$$

where[13]

$$\mathcal{Z}_1 = \exp\left[-\frac{\beta}{2}\mathrm{Tr}(M_1)\right] \left|\det[1 + \exp(\beta N_1)]\right|^{1/2} \tag{44}$$

is the partition function of the system in which all $\eta_{ij}=1$ (the anyon free system). In Eq. (43), the operator $\mathbf{E}$ projects out of $\mathcal{Z}_1$ only those terms that are even (hence, the symbol $\mathbf{E}$) in all exchange constants $\{J_{ij}\}$. The matrices N_1 and M_1 depend on the constants $\{J_{ij}\}$ as detailed in Ref. 13. The same conclusion follows from the expansion Eq. (37).

We note that for the particular set of exchange constants $J_{i,i+\hat{e}_{\mu}} = J_{\mu}$, (nearest-neighbor J_{ij}), Eq. (43) is indeed equal to

$$\mathcal{Z} = 2^{N_h + 1} \mathbf{E}^{z}[\mathcal{Z}_1], \tag{45}$$

where $\mathbf{E}^z$ projects out of $\mathcal{Z}_1$ only the terms that have all of the powers of $J_{i,i+\hat{e}_z}$ being even. In order to cover all of the topological sectors $I_{h_{\alpha}} = \pm 1$, in each hexagon h_{α} it suffices to allow $\eta_{ij} = \pm 1$ on all of the vertical bonds (parallel to the z direction), and $\eta_{ij}=1$ on all other bonds (parallel to the x or y directions).[24]

It is worth emphasizing that different topological sectors $\{I_{h_{\alpha}}\}$ lead to different $\mathcal{Z}_{\eta}$ (and thus to different spectra as they indeed must). It is only after performing the trace in Eq. (38) that the common even powered terms are pulled out. These terms are the same in all $\{\eta_{ij}\}$ assignments.

III. VECTOR-EXCHANGE LATTICE MODEL

A. Motivation

Consider the Lagrangian density of fermions coupled to a vector gauge field A_a with $a=0,1,2,3$. In a U(1) theory, the Lagrangian density describing the minimal coupling of fermions to the gauge field is given by

$$\mathcal{L}_{\mathrm{min}} = \bar{\psi}(i\gamma^{\mu}\partial_a - \gamma^{\mu}A_a)\psi, \tag{46}$$

where $\gamma^{\mu} = (\gamma^{\mu})^{\dagger}$ are the Dirac matrices. Within the U(1) theory, $A_{a=0}$ is the scalar potential and $A_{a=1,2,3}$ are the spatial components of the vector potential $\vec{A}$. The minimal-coupling term of Eq. (46) is augmented by a gauge-only term $(\frac{1}{4}F_{ab}F^{ab}$ with $F_{ab} = \partial_a A_b - \partial_b A_a)$. In the electroweak theory

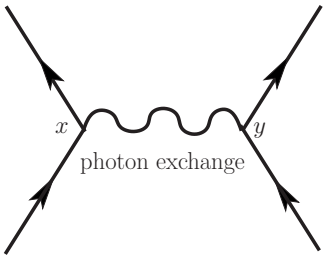

FIG. 2. Vector exchange between two fermions. To lowest (quadratic) order in the gauge field, a bilinear between the Dirac matrices: $\bar{\psi}(x)\gamma^a\psi(x)D_{ab}(x,y)\bar{\psi}(y)\gamma^b\psi(y)$ results. For a vector-exchange (*photon*) propagator $D_{ab}(x,y)$ set to be $J_{x,y}$ on a lattice, the resulting system is precisely of the form of the vector-exchange model of Eq. (50).

$[\mathrm{SU}(2)\times\mathrm{U}(1)]$, the A_a in Eq. (46) is replaced by $(A_a - V_a)$ with the weak parity breaking field V_a.

Although the Lagrangian is quadratic in the fermion fields ψ, it is definitely not a simple quadratic form that can be exactly integrated out. This is due to the linear coupling to A_a. The theory contains both *free* quadratic terms (e.g., those in ψ alone) and terms of the form $\bar{\psi}\gamma^a A_a\psi$. These terms give rise to interactions such as the lowest-order exchange term shown in Fig. 2. The coupling to the A_a gauge field gives rise to the usual Coulomb interaction between fermions.

The lowest-order interaction terms are those formed by two vertices as above. The slanted lines depict the fermions (ψ) while the horizontal wavy line represents the *photon* propagator (D_{ab})–the propagator for the fields A_a. Integrating out the gauge field gives rise to the usual Coulomb exchange (depicted in Fig. 2)

$$\bar{\psi}(x)\gamma^a\psi(x)D_{ab}(x,y)\bar{\psi}(y)\gamma^b\psi(y) \tag{47}$$

with

$$D_{ab}(x,y) = \langle A_a(x)A_b(y)\rangle \tag{48}$$

the Coulomb propagator. The same formalism albeit with more indices applies to other vector gauges (e.g., the electroweak one). In the nonrelativistic limit, the density-density interaction [the $\bar{\psi}(x)\gamma^0\psi(x)\bar{\psi}(y)\gamma^0\psi(y)$] piece becomes important. That is, the D_{00} propagator becomes dominant for nonrelativistic particles.

The lattice gauge action for the fermions resulting from integrating out the vector gauges A_a is not usually investigated in lattice gauge theory calculations. It is correct but this is not the standard point of departure for lattice gauge calculations. What is typically done is to write terms of the form

$$\bar{\psi}(x)\gamma^a\psi(x+\hat{e}_a)\exp[iA_{x,x+\hat{e}_a}] \tag{49}$$

with $A_{x,x+\hat{e}_a}$ as the line-integrated lattice gauge (3+1) vector potential between nearest-neighbor sites.

What we do in the following affords another way of investigating general minimally coupled actions. When integrating out the A_a fields, we generate precisely interactions of the $\gamma\gamma$ type of Eq. (47) with Eq. (48). In what follows, we will investigate a simple lattice rendition of such vector-exchange system in which we set an exchange coupling between γ matrices to be of amplitude $J_{ij}\equiv D(i,j)$ with i and j denoting lattice sites. In order to have a simple algebra, we will choose the propagator D to vanish unless i and j are nearest-neighbor sites. Furthermore, similar to Kitaev's honeycomb model, we consider the situation wherein the physical location of the sites i and j determines what components of the γ matrices at the two sites i and j are coupled in the Hamiltonian.

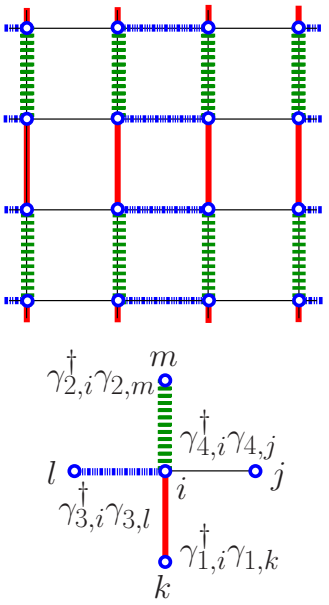

FIG. 3. (Color online) The bonds in the system of Eq. (50). At each vertex i there are four different types of bonds corresponding to $\gamma_{a,i}\gamma_{a,j}$ interactions with different $a=1,2,3,4$. (There are four different types of vertices.) The algebra of the bonds is trivial: bonds that share a site anticommute, disjoint bonds commute, and the square of any bond is one. Consequently the spectrum of the model can be immediately determined.

B. Exact solution of the vector-exchange model

A simple square lattice model that captures the fermionic vector exchange is given by

$$H = \sum_{\langle ij\rangle} J_{ij}\gamma_{a,i}\gamma_{a,j}. \tag{50}$$

The geometry of the lattice is shown in Fig. 3. The γ matrix index a for a given bond in Eq. (50) is fixed by the sites i and j. Here and throughout, we set the lattice constant to be one.

The γ matrices satisfy the algebra

$$\{\gamma_{a,i},\gamma_{b,i}\} = 2\delta_{ab}, \quad [\gamma_{a,i},\gamma_{b,j}] = 0, \quad i\neq j. \tag{51}$$

The Hilbert space on which H acts on is, for a lattice of N_s sites, of dimension 4^{N_s}. The algebra of the bonds $\gamma_{a,i}\gamma_{a,j}$ is familiar: it has the same simple characteristics of the bond algebra in Kitaev's honeycomb model. These algebraic relations do not change in projection to a state of fixed topological charge sector

$$\hat{P}_{\square}(\gamma_{a,i}\gamma_{a,j})\hat{P}_{\square}. \tag{52}$$

We define the projector $P_{\square}$ to a topological sector by

$$\hat{P}_{\square} = \frac{\mathbb{1} + \tilde{I}_{\square} I_{\square}}{2}, \tag{53}$$

with

$$I_{\square} = \prod_{\langle ij \rangle \in \square} \gamma_{a,i}\gamma_{a,j}. \tag{54}$$

As in Eq. (28), $\tilde{I}_{\square}$ are c numbers: $\tilde{I}_{\square} = \pm 1$, and products of bonds around a plaquette $I_{\square}$ commute with the Hamiltonian ($[I_{\square}, H]=0$), and among themselves ($[I_{\square}, I_{\square'}]=0$). The operators of Eq. (54) constitute local (gauge) symmetries. The origin of the commutation relations is that at each vertex we have bonds of different γ matrix flavors. Similar to the situation in Kitaev's honeycomb model, all bonds commute with the anyon charge operators of Eq. (54), and

$$I_{\square}^2 = 1. \tag{55}$$

The gauge symmetries $\{I_{\square}\}$ allow decomposition of the total Hilbert space into orthogonal subspaces of equal dimensionality. We divide the Hilbert space into equal sectors spanned by $\{\tilde{I}_{\square}\}$. There are 2^{N_s-1} such sectors as the eigenvalues of $I_{\square}$, for each of the N_s plaquettes $\square$, can attain one of two values (± 1), and satisfy only one global constraint on a torus

$$\prod_{\square} I_{\square} = 1. \tag{56}$$

As $\{\tilde{I}_{\square}\}$ are good quantum numbers, we may diagonalize the Hamiltonian in a Hilbert space of dimension $4^{N_s}/2^{N_s-1} = 2^{N_s+1}$. Similar to our solution of the Kitaev's honeycomb model, we may then work with the representation of the bonds as the product of two fermions.

Within each anyon charge sector, the Hamiltonian is of the form of Eq. (34) but on different size spaces. We can now introduce N_s spinless fermion variables $\{d_i\}$ on the Hilbert space of size 2^{N_s} by setting the bonds to be

$$\bar{A}_{ij} = i(d_i + d_i^{\dagger})(d_j + d_j^{\dagger}). \tag{57}$$

We thus arrive at a Fermi bilinear that is trivially diagonalizable

$$H = i\sum_{\langle ij \rangle} \eta_{ij} J_{ij} (d_i + d_i^{\dagger})(d_j + d_j^{\dagger}). \tag{58}$$

In Eq. (58), we maintain the directionality that we employed throughout in constructing the bond algebra in the case of Kitaev's honeycomb model: $j - i = \hat{e}_x$ or $\hat{e}_y$. A trivial but important feature of Eq. (58) is that the spectrum is symmetric about zero. This is so as there is a symmetry $\bar{A}_{ij} \to -\bar{A}_{ij}$ in the representation chosen in Eq. (57).

The dimension of the Hilbert space is the same as that of the product of all plaquette charges $\tilde{I}_{\square} = \Pi_{\langle ij \rangle \in \square} \eta_{ij}$. Due to Eq. (56), there are $(N_s - 1)$ such Ising-type operators each with eigenvalues ± 1. These lead to 2^{N_s-1} sectors. This number is to be multiplied by the size of the Hilbert space spanned by the N_s fermions (2^{N_s}) and further multiplied by a degeneracy factor of two (associated with the degeneracy $\bar{A}_{ij} \to -\bar{A}_{ij}$). Fixed anyon charges enable 2^{N_s} possible configurations (redundant) of η_{ij} that give rise to the same original Hamiltonian when projected onto a sector of fixed $\{I_{\square}\}$. These configurations of η_{ij} are related to each other by local Ising gauge transformations on the square lattice. That is, with arbitrary $\tau_i = \pm 1$ at any lattice site i, the local gauge transformation

$$\eta_{ij} \to \tau_i \eta_{ij} \tau_j \tag{59}$$

leaves $\tilde{I}_{\square}$ invariant.

Written longhand, the 4^{N_s}-dimensional Hilbert space spanned by the γ matrices decomposes as follows:

$$\begin{aligned} 4^{N_s} = [&2^{N_s-1}(\text{number of sectors } \{\tilde{I}_{\square}\}) \\ &\times 2^{N_s}(\text{Hilbert space spanned by } N_s \text{ fermions}) \\ &\times 2(\text{remaining degeneracy of each state})]. \end{aligned} \tag{60}$$

There is a degeneracy factor of $2 = 4^{N_s}/(2^{N_s-1} \times 2^{N_s})$ that remains after invoking the representation of Eq. (57) in the space of size 2^{N_s}. This is similar to the degeneracy factor of two in Sec. II A. The Hamiltonian of Eq. (58) is nothing but a tight-binding Hamiltonian augmented by pairing terms (an element of the so($2N_s$) algebra) on which we may apply a Bogoliubov transformation similar to Ref. 24 which was defined on the square lattice. The solution to Eq. (58) can be immediately written down. For J_{ij} equal to J_x or J_y for sites i and j separated by one lattice constant along the x or y directions, respectively, i.e., $J_{ij} = (J_x \delta_{|i_x - j_x|,1} + J_y \delta_{|i_y - j_y|,1})$, in the sector $\eta_{ij} = 1$ (corresponding to the sector $\tilde{I}_{\square} = 1$), we have on Fourier transforming,

$$H = i\left[\sum_{\vec{k}} q_{\vec{k}} d_{\vec{k}}^{\dagger} d_{-\vec{k}}^{\dagger} + q_{-\vec{k}} d_{\vec{k}} d_{-\vec{k}}\right] - \sum_{\vec{k}} p_{\vec{k}} d_{\vec{k}}^{\dagger} d_{\vec{k}}, \tag{61}$$

with $q_{\vec{k}} \equiv [J_x e^{ik_x} + J_y e^{ik_y}]$ and $p_{\vec{k}} \equiv 2(J_x \sin k_x + J_y \sin k_y)$. A Bogoliubov transformation gives the quasiparticle spectrum

$$E_{\vec{k}} = 0, \pm 2p_{\vec{k}}, \tag{62}$$

with the zero eigenvalue being doubly degenerate. It is noteworthy how easy and immediate the solution is.[27]

C. String correlation functions from symmetries

The only correlators that can obtain a finite expectation value at zero and finite temperatures must, by Elitzur's theorem,[28] be invariant under all local symmetries. In Ref. 24, this was invoked to show how symmetries only allowed for string correlators of the form of Eq. (31) to have nonvanishing expectation values at finite and zero temperatures. These symmetries' only conditions did not need to invoke the particular Majorana fermion representation of the Kitaev model that was used in Ref. 29 to study the zero-temperature correlation functions. The considerations of Ref. 24 for Kitaev's model can be replicated mutatits muntandis for the vector-exchange model. The local symmetries of Eq. (54) allow only for string correlators (whether open or closed) to

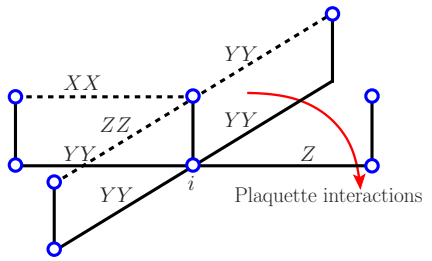

FIG. 4. (Color online) A square lattice bilayer of a $S=1/2$ system that represents Eq. (50). Inserting Eq. (64), we find the interactions schematically depicted above. The two layers are indexed by the first of the subscripts (1 or 2) in Eq. (64).

attain a finite expectation value. This applies for both the ground-state configuration as well as the more physically pertinent case of all finite temperatures, $T>0$. The zero-temperature selection rule was also noted by Ref. 30. Closed loops of the form of Eq. (31) that span the entire system correspond to additional symmetries of the system. Relying on similar symmetry conditions as in Ref. 9, it is seen that the only orders that can exist are of a nonlocal nature.

D. Isomorphic spin models

We now discuss two spin representations of our exactly solvable vector-exchange model: (i) a spin $S=1/2$ system on a square lattice bilayer and (ii) a spin $S=3/2$ system on a square lattice. This latter variant was very recently also discussed by Refs. 30 and 31.

1. Spin 1/2 model on a square lattice bilayer

A possible representation of Eq. (50) is obtained at by setting

$$\gamma_\mu = \begin{pmatrix} 0 & (-i\sigma^\mu) \\ (i\sigma^\mu) & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}, \tag{63}$$

with $\mu=1,2,3$, σ^μ the Pauli matrices, and $\mathbb{1}_2$ the two-dimensional unit matrix. The γ matrices can arise from the tensor product of two $S=1/2$ spins at each lattice site.

$$\gamma_{1,i} = \sigma^y_{1,i}\sigma^x_{2,i},$$

$$\gamma_{2,i} = \sigma^y_{1,i}\sigma^y_{2,i},$$

$$\gamma_{3,i} = \sigma^y_{1,i}\sigma^z_{2,i},$$

$$\gamma_{4,i} = \sigma^z_{1,i}. \tag{64}$$

A possible lattice topology realization for this is that of a square lattice bilayer shown in Fig. 4. The subscript α in $\sigma^\mu_{\alpha,i}$ is the bilayer index. Inserting Eq. (64) into Eq. (50) leads to a spin $S=1/2$ Hamiltonian on a square lattice bilayer.

2. Spin 3/2 model on a square lattice

Another representation of the model of Eq. (50) is in terms of $S=3/2$ spins that reproduces the results of Refs. 30 and 31 for their $J_5=0$. The *spin liquid* character of these systems is inherited from Kitaev's honeycomb model.

$$S^x = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \tag{65}$$

$$S^y = \begin{pmatrix} 0 & -i\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{pmatrix}, \tag{66}$$

and

$$S^z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}. \tag{67}$$

In terms of two spins of size $S=1/2$,

$$S^x = \frac{\sqrt{3}}{2}\sigma^x_2 + \frac{1}{2}(\sigma^x_1\sigma^x_2 + \sigma^y_1\sigma^y_2),$$

$$S^y = \frac{\sqrt{3}}{2}\sigma^y_2 + \frac{1}{2}(\sigma^y_1\sigma^x_2 - \sigma^x_1\sigma^y_2),$$

$$S^z = \sigma^z_1 + \frac{1}{2}\sigma^z_2,$$

$$(S^x)^2 = \frac{\sqrt{3}}{2}\sigma^x_1 - \frac{1}{2}\sigma^z_1\sigma^z_2 + \frac{5}{4},$$

$$(S^y)^2 = -\frac{\sqrt{3}}{2}\sigma^x_1 - \frac{1}{2}\sigma^z_1\sigma^z_2 + \frac{5}{4},$$

$$(S^z)^2 = \sigma^z_1\sigma^z_2 + \frac{5}{4},$$

$$\{S^x, S^y\} = \sqrt{3}\sigma_1^y,$$

$$\{S^y, S^z\} = \sqrt{3}\sigma_1^z\sigma_2^y,$$

$$\{S^x, S^z\} = \sqrt{3}\sigma_1^z\sigma_2^x. \tag{68}$$

We can represent the γ matrices by

$$\gamma_1 = \sigma_1^z\sigma_2^y = \frac{1}{\sqrt{3}}\{S^y, S^z\},$$

$$\gamma_2 = \sigma_1^z\sigma_2^x = \frac{1}{\sqrt{3}}\{S^x, S^z\},$$

$$\gamma_3 = \sigma_1^y = \frac{1}{\sqrt{3}}\{S^x, S^y\},$$

$$\gamma_4 = \sigma_1^x = \frac{1}{\sqrt{3}}((S^x)^2 - (S^y)^2),$$

$$\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \sigma_1^z\sigma_2^z = (S^z)^2 - \frac{5}{4}. \tag{69}$$

IV. CLIFFORD ALGEBRAIC MODELS

The commonality of all these exactly solvable models is the presence of degrees of freedom that satisfy a Clifford algebra on-site

$$\{\gamma_{a,i}, \gamma_{b,i}\} = 2\delta_{ab}, \tag{70}$$

and a commutative algebra off-site

$$[\gamma_{a,i}, \gamma_{b,j}] = 0, \quad i \neq j, \tag{71}$$

with $a, b = 1, \ldots, p$. The exactly solvable Hamiltonians are then written as linear combinations of quadratic products of these γ matrices. Regardless of the dimension of the representation of the γ matrices, the Hamiltonian is always an element of so$(2N_s)$ (in the examples worked out in this paper), and thus, a GMFH.

From the viewpoint of lattice connectivity, notice a fundamental difference between Kitaev's honeycomb and the $\gamma\gamma$ (vector-exchange) models. The coordination of the honeycomb lattice is $z_h = 3$, while the one for the $\gamma\gamma$ lattice is $z_\gamma = 4$. This is the reason why one needs $p = 3$ anticommuting (Pauli) matrices in the first case, while $p = 4$ anticommuting (γ) matrices are needed in the second model. The relation between p and the dimension of the matrix representation of γ is the following: when $p = 2q$ or $p = 2q+1$, the matrix representation can be of dimension 2^q. This is the reason why Kitaev used Pauli matrices ($q = 1$) in his honeycomb model, while we had used Dirac matrices ($q = 2$) in the vector-exchange model.

It is indeed obvious how to generalize these ideas to generate new exactly solvable models of the so$(2N)$ type in arbitrary dimensions and for arbitrary lattice coordination. The idea consists in writing bond operators which are quadratic products of Clifford operators which are anticommuting on the same *lattice site*. The cardinal p of that set of operators will define the z of the lattice (its connectivity). For instance, suppose we want to have a lattice with $z = 5$. Then, a Shastry-Sutherland-like connectivity lattice will do the job.[32] Now, write down a Hamiltonian which is a linear combination of bilinears of five anticommuting γ matrices that act upon a Hilbert space of dimension 4^{N_s}. This model will be exactly solvable. In this way, we can construct a new model in a cubic $z = 6$ lattice with $p = 6$ anticommuting matrices, or as we show now a triangular $z = 6$ lattice model with γ matrices of dimension $2^3 \times 2^3$.

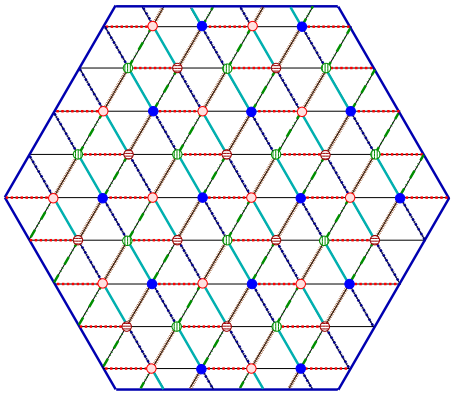

FIG. 5. (Color online) A triangular lattice Clifford model that is exactly solvable. In each vertex there is a degree of freedom of dimension 2^3 (and a Hilbert space of dimension 2^{3N_s}). There are four different types of vertices and six different types of bonds $\gamma_{a,i}\gamma_{a,j}$.

Consider the $p = 6$ γ matrices

$$\gamma_{1,i} = \sigma_{1,i}^y\sigma_{2,i}^x,$$

$$\gamma_{2,i} = \sigma_{1,i}^y\sigma_{2,i}^y,$$

$$\gamma_{3,i} = \sigma_{1,i}^y\sigma_{2,i}^z,$$

$$\gamma_{4,i} = \sigma_{1,i}^z\sigma_{3,i}^x,$$

$$\gamma_{5,i} = \sigma_{1,i}^z\sigma_{3,i}^y,$$

$$\gamma_{6,i} = \sigma_{1,i}^z\sigma_{3,i}^z, \tag{72}$$

which form an on-site Clifford algebra. The model Hamiltonian

$$H = \sum_{\langle ij \rangle} J_{ij}\gamma_{a,i}\gamma_{a,j}, \tag{73}$$

whose lattice geometry is shown in Fig. 5, is exactly solvable.

This model can also represent a trilayer system with plaquette interactions.

One can indeed realize that there is nothing special about the Lie algebra so$(2N)$. One can consider models whose bond algebra forms any other semisimple Lie algebra, such as so$(2N+1)$ where there are nonlinear Bogoliubov transfor-

mations that diagonalize the problem. It is important though, that the number of gauge symmetries is enough to allow for a simple oscillator realization of the bonds. Otherwise, there is always the possibility to use the Jacobi algorithm[5] to numerically diagonalize the problem.

V. CONCLUSIONS

The thesis of the current work is that even though the solution to many problems is hard, by disregarding the explicit microscopic degrees of freedom and focusing solely on the algebraic relations that the bond variables satisfy in a Hilbert space of a fixed dimensionality, we may map one initially seemingly hard problem onto another problem whose solution is easier. This mapping does not rely on explicit real-space forms for the transformations (although these can be written down in some cases). Nor does it rely on enlarging the Hilbert space and then making a projection onto a physical sector as in Ref. 7. It is important to emphasize though, that the dimension of the *representation* of the algebra is of crucial importance—not only the algebra and set of constraints itself.

The explicit real-space mappings—no matter how complicated their forms are—are irrelevant. The spectra and all nonvanishing correlators may be determined from the algebra alone. [See Ref. 9 for a derivation of all correlators in Kitaev's toric code model by this method]. In the current work, we illustrated how the energy spectra may be determined. The partition function and the density of states associated with the spectrum are related by a Laplace transform. It may be easily seen also from the partition functions themselves that if two systems display the same bond algebra on a space of the same representation then their spectra are identical.

In the case of generic GMFHs, the Jacobi method always enables a solution of its spectrum with polynomial complexity.[5] There are situations, such as the model examples presented in this paper, where we can determine the spectrum of certain sectors (Hilbert subspaces) in closed form, i.e., by quadrature. This will happen whenever the effective matrices that need to be diagonalized have dimension smaller or equal to 4×4 (because of Galois theory). The decomposition of the Hilbert space into these individual decoupled subspaces is rooted in the existence of *local (gauge) symmetries*. Within each subspace, there is an *oscillator realization* (e.g., in terms of Majorana fermions) of the bond algebra. It is important to emphasize that exact solvability does not imply that we can compute the density of states, and thus, the partition function, with polynomial complexity. Kitaev's honeycomb model is an example of a system whose energy eigenvalues can be determined with polynomial complexity but whose total partition function cannot since its density of states is not determined with the same complexity. By contrast, Kitaev's toric code model constitutes an example of a system where not only the spectrum but also its partition function is exactly solvable.[8,9]

As we illustrated in this work, there are many applications of the bond algebraic methodology that we introduced here. Some systems that are analyzed anew by investigators can be related via bond algebraic mappings to models that have already been hitherto investigated. As we will illustrate elsewhere, a powerful offshoot of our method enables us to analyze and derive new dualities in physical systems.[33]

Our bond algebraic framework, including the vector-exchange model, was conceived in 2007. We first applied our technique to the solution of Kitaev's toric code model.[8,9] Later on a physical model whose exact solution was enabled by our bond algebra mapping appeared in Ref. 1. During the time in which the current work summarizing our general approach was prepared, three works[30,31,34] that study variants of the vector-exchange model that we present here appeared. In particular, related to our representation in Sec. III D 2,[30,31] present a spin $S=3/2$ vector-exchange model on a decorated square lattice.

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[10] Although finding an eigenvalue is easy and hence, the problem is exactly solvable in the sense defined above, it is worth emphasizing that it is far harder to determine the complete density of states (i.e., the degeneracy of each of the eigenvalues) or the Laplace transform of the density of states—the partition function. Indeed, determining the asymptotic thermodynamic limit form of the partition function of the Ising model on lattices of spatial dimension larger than two is a long-standing intractable problem. Within the bond algebraic formulation, the crux of the difficulty in determining the partition function can be tracked down in the constraints that the bonds satisfy

$$\prod_{\langle ij\rangle\in\square} b_{ij} = 1 .$$

The product in this equation is that of the bonds b_{ij} around any small elementary closed-loop (plaquette) $\square$ of the lattice.

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[15] It is important to note that all of the steps invoked in the high-temperature expansion discussed below rely only on the operator algebra and the dimension of its representation. First, note that $\exp[\beta(A_s+B_p)]=\exp(\beta A_s)\exp(\beta B_p)$. Then, the partition function is

$$\mathcal{Z}=(\cosh\beta)^{2N_s}\mathrm{Tr}_{\{\sigma\}}\sum_G(1+(\tanh\beta)^{|S|}\prod_{s\in G}A_s)$$

$$\times(1+(\tanh\beta)^{|P|}\prod_{p\in G}B_p),$$

with G all graphs on the lattice (subset of lattice sites). The graph G spans $|S|$ sites and $|P|$ plaquettes. As $A_s=\Pi_i\sigma_{is}^x$ and $B_p=\Pi_{ij\in p}\sigma_{ij}^z$, and as the trace of any Pauli-matrix σ_{ij}^μ raised to an odd power is zero, the only terms surviving in the sum of the partition function are those with graphs G such that each lattice link appears in an even number of star operators (i.e., zero or two) and an even number of plaquettes (zero or two). The trace over these nonvanishing terms is 2^{2N_s}, as there are 2^{2N_s} possible bond configurations (and a total of $2N_s$ bonds) and σ_{ij}^μ raised to an even power is the identity operator. The only graphs G on a torus that satisfy this condition are those in which (i) G is the empty set, (ii) G contains all stars in the lattice, (iii) G contains all plaquettes in the lattice, or (iv) G contains all stars and plaquettes in the lattice. These give rise to finite terms as the products over the entire torus, $\Pi_s A_s=1$ and $\Pi_p B_p=1$, by virtue of the topology (each bond on a square lattice on a torus appears in an even number of plaquettes or stars). Summing up these four graphs we have, Eq. (19).

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[20] As a curiosity, it is worth noting that our bond algebraic mapping gives rise to a crossover for finite-size systems. As we showed by the application of bond algebras, Kitaev's toric code model is isomorphic to two decoupled Ising chains. The correlation length within an Ising chain of exchange constant $J=1$ as in Eq. (18) is given by

$$\xi=\frac{1}{\ln(\coth\beta)}.$$

When $\xi\gg N_s$, the system may appear to be ordered while for $\xi\ll N_s$, the system is disordered across the chain. When $\beta\gg 1$, the function

$$\frac{1}{\ln(\coth\beta)}\to\frac{e^{2\beta}}{2}.$$

Consequently, the crossover temperature at which $\xi(T)\simeq N_s$ is given by

$$k_BT_{\mathrm{cross}}=\frac{2}{\ln(2N_s)}.$$

For $T<T_{\mathrm{cross}}$ the system may support instantaneous local order across a finite-size system. As $N_s\to\infty$, the crossover temperature tends to zero. The existence of a crossover temperature with a $1/\ln(2N_s)$ scaling that we derived here ($k_BT_{\mathrm{cross}}=\frac{2}{\ln(2N_s)}$) from our old bond algebra mapping to a one-dimensional system (Refs. 8 and 9) coincides with that seen in the entanglement entropy (Ref. 21). Related scalings for mutual information were derived in Ref. 22.

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[27] As a trivial aside, apart from the standard Bogoliubov transformation, the spectrum can also be attained by a nonstandard representation of the (Fermi) bond algebra. Towards this end, we examine all wave numbers $\vec{k}_1$ that are not half a reciprocal vector that have a positive x component $k_{1x}>0$. For spinless fermions, each of the wave numbers $\vec{k}_1$ and $(-\vec{k}_1)$ can be occupied or unoccupied. Thus, the Hilbert space associated with each wave number pair is $2^2=4$ dimensional. A trivial representation that preserves the Fermi algebra is achieved by the following mapping $[\sigma_i^\pm=(\sigma_i^x\pm i\sigma_i^y)/2]$:

$$d_{\vec{k}_1}\to\sigma_1^-,\quad d_{\vec{k}_1}^\dagger\to\sigma_1^+,$$

$$d_{-\vec{k}_1}\to\sigma_1^z\sigma_2^-,\quad d_{-\vec{k}_1}^\dagger\to\sigma_1^z\sigma_2^+.$$

In the preceding equation, we write the Fermionic operators for each wave number pair $|n_{\vec{k}_1},n_{-\vec{k}_1}\rangle$ in the four-dimensional (4D) basis spanned by two $S=1/2$ spins ($|\sigma_1^z\sigma_2^z\rangle$). Products involving spin operators at sites 1 and 2 in the above-mentioned equation are to be understood as a tensor product of the operators at both sites. Writing the sum in Eq. (61) as a sum over half of the wave number $\vec{k}_1$ and employing the four-dimensional representation for the Hamiltonian within each of the decoupled sectors ($\vec{k}_1$, $-\vec{k}_1$), we find that the eigenvalues are $\epsilon_{k_1}=\pm p_{\vec{k}_1}$. Within each wave number doublet this gives (as it must, up to a trivial additive uniform shift) the result of the standard Bogoliubov transformation for a spinless Fermi bilinear. [In the comparison between the two forms (the eigenvalues $\epsilon_{\vec{k}_1}$ for the wave numbers $\vec{k}_1$ with $k_{1x}>0$ and the quasiparticle energies $E_{\vec{k}}$ for all wave numbers $\vec{k}$ in the first Brillouin zone) it should be borne in mind that $p_{-\vec{k}_1}=-p_{\vec{k}_1}$].

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